THE ŠILOV BOUNDARY OF $M(G)$

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$M = M(G)$ is the convolution algebra of bounded measures on the locally compact abelian (L.C.A.) group G [2, p. 328], Δ is its maximal ideal space and ∂M its Šilov boundary [2, p. 132]. Taylor [6] has shown that for $G = \boldsymbol{Z}(2) \times \boldsymbol{Z}(2) \times \cdots$ we have $\partial M \neq \Delta$. We shall show how Taylor's method can be extended to give this result for any L.C.A. nondiscrete group G.

Taylor's method depends on his theory of convolution measure algebras. We shall use Šreĭder's theory of generalized characters [4] as it is essentially equivalent and is more widely known. Accordingly we consider Δ as a set of generalized characters. The method depends on constructing a $\mu \in M(G)$ with the surprising property that for any $\chi \in \Delta$, χ_μ is equal (almost everywhere with respect to μ) to a multiple of a continuous character. In §1 we show how the existence of such a μ implies $\partial M \neq \Delta$. The construction of μ is carried out in four special groups or types of group. The general pattern of the construction is the same in each case and that part of the proof which is common to each case is carried out in §2. The particular considerations which complete the proof are given for an infinite product of cyclic groups in §3, for the additive group of p-adic integers in §4 and for $\boldsymbol{R}$ and T, the additive group of reals modulo 1, in §5. In §6 we show how the existence of a suitable μ on a closed subgroup implies its existence on G and how the property $\partial M \neq \Delta$ for G can be deduced from the same property for G/H, H a compact subgroup. In §7 we show that the general result can be deduced from the previous sections.

1. **Reduction to the question of the existence of a suitable measure.** Let G be a L.C.A. group and $\mu \in M(G)$. For $a \in \boldsymbol{C}$, $\boldsymbol{a}$ denotes the function on G with constant value a. We put

$$D = \{a; a \in \boldsymbol{C}, \boldsymbol{a} \in (G^\wedge)^-\}$$

where $^-$ indicates $\sigma(L^\infty(\mu), L^1(\mu))$ closure.

1.1. *If $\mu \in M(G)$ such that*

(i) *for any $\chi \in \Delta$ there is $a \in \boldsymbol{C}$ and $\Psi \in G^\wedge$ with $\chi_\mu = a\Psi$, μ almost everywhere in G,*

(ii) *D contains numbers a with $0 < |a| < 1$ but $\{|a|; a \in D\}$ does not contain the whole of $(0, 1)$, then $\partial M \neq \Delta$.*

Proof. First of all we show that any constant nonzero function $\boldsymbol{b}$ in $\{\chi_\mu; \chi \in \partial M\}$ has $|b| = |a|$ for some $a \in D$. Suppose $\chi \in \partial M$ with $\chi_\mu = \boldsymbol{b}$.

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Theorem 3.3 of [5], restated in terms of generalized characters, implies that χ is a limit (in Δ) of a net χ_α of generalized characters with $|(\chi_\alpha)_\sigma(g)|=0$ or 1 for σ almost all $g \in G$ for each $\sigma \in M(G)$. Thus $(\chi_\alpha)_\mu \to \boldsymbol{b}$ in $\sigma(L^\infty(\mu), L^1(\mu))$. By condition (i) we can write $(\chi_\alpha)_\mu = a_\alpha \phi_\alpha$ where $\phi_\alpha \in G^\wedge$, $a_\alpha \in C$ and we have $|a_\alpha|\ |\phi_\alpha| = |a_\alpha| = 0$ or 1. By transferring to a subnet if necessary we can assume that $c = \lim a_\alpha$ exists. Clearly $|c|=1$ and $c\phi_\alpha \to \boldsymbol{b}$. Thus $a = bc^{-1} \in D$ and $|a|=|b|$.

Now if $a \in D$ with $0<|a|<1$ and χ_β is a net in $G^\wedge$ with $\chi_\beta \to \boldsymbol{a}$ in $\sigma(L^\infty(\mu), L^1(\mu))$, by considering $G^\wedge$ embedded in Δ in the usual way, in which case $G^\wedge \subset \partial M$, we can find a subnet χ_α of χ_β converging to a point χ in Δ. Clearly $\chi \in \partial M$ and $\chi_\mu = \boldsymbol{a}$. Since, for any z with $\operatorname{Re} z>0$, $|\chi|^z$ is a generalized character and $(|\chi|^z)_\mu = |a|^z\mathbf{1}$ where $|a|^z$ takes all values in $0<|w|<1$ as z ranges over $\operatorname{Re} z>0$ we see that $\{|a|; \boldsymbol{a}=\chi_\mu \text{ for some } \chi \in \Delta\}$ contains the whole of $(0, 1)$ whereas $\{|a|; \boldsymbol{a}=\chi_\mu \text{ for some } \chi \in \partial M\} = \{|a|; a \in D\}$ does not. Thus $\Delta \neq \partial M$.

2. **The general construction of μ.** We now suppose that we have the following: $V_0, V_1, \ldots$, a basic system of compact neighborhoods of 0 in G with $V_i \supset V_{i+1} + V_{i+1}$; $m_1, m_2, \ldots$, positive integers; x_{ij}, $i=1, 2, \ldots, j=0, 1, \ldots, m_i-1$ such that $x_{i0}=0$, $x_{ij} \in V_{i-1}$ and V_{i-1} is the union of the disjoint sets $x_{ij}+V_i$, $j=0, \ldots, m_i-1$.

We put

$$X_i = \{x_{ij}; j = 0, \ldots, m_i-1\}, \quad Y_i = X_1+X_2+\cdots+X_i,$$

$$M_i = 2[m_i(m_i+1)]^{-1}, \quad \delta_i = M_i \sum_j (m_i-j)\delta(x_{ij}),$$

$$\mu = \mathop{\ast}_{i=1}^{\infty} \delta_i, \quad \mu_n = \mathop{\ast}_{i=n+1}^{\infty} \delta_i$$

where $\delta(x)$ is the point mass at x. The infinite convolutions converge by [7, Lemma 8] and $\mu = [\sum_{y\in Y_i} \mu(y+V_i)\delta(y)] * \mu_i$. It is important for §5 that we only use the compactness of the V_i to show that μ_n is concentrated on V_n. We shall reduce the problem of showing that μ satisfies the hypothesis of 1.1 to the consideration of certain limits.

Let S be the (not necessarily closed) countable subgroup of G generated by all the x_{ij} and let ν be a bounded measure obtained by placing point masses at each of the points of S. Put $\sigma=\mu+\nu$. Then for $\chi \in \Delta$, $\chi_\sigma(s+t)=\chi_\sigma(s)\chi_\sigma(t)$ for $\sigma\times\sigma$ almost all (s, t) in $G\times G$ so that, for each $s \in S$, $\chi_\sigma(s+t)=\chi_\sigma(s)\chi_\sigma(t)$ for σ almost all $t \in G$. In particular χ_σ is a character on S. Note also that $\chi_\sigma=\chi_\mu$ μ almost everywhere.

For $y \in Y_i$, $c(i, y)$ is the characteristic function of $y+V_i$ and C_i is the linear span of the $c(i, y)$. Since for each i the sets $y+V_i$ cover V_0 and the V_i form a base of the neighborhood system of 0, we can find, for each continuous function g on G and each i, a $g_i \in C_i$ such that $g_i \to g$ uniformly on V_0. Since V_0 contains the support of μ and $\chi_\sigma \in L^\infty(\mu) \subset L^1(\mu)$ we can find, for each i, an $f_i \in C_i$ with $\|f_i-\chi_\sigma\|_1 \to 0$ ($L^1(\mu)$ norm). As the C_i are finite dimensional we can suppose the f_i chosen so that

$\|f_i-\chi_\sigma\|_1$ is the least possible. Suppose $f_i=\sum a(i,y)c(i,y)$ where $a(i,y)\in C$ and the summation is over $y\in Y_i$. We have

$$\|f_i-\chi_\sigma\|_1 = \sum_{y\in Y_i}\int_{y+V_i}|a(i,y)-\chi_\sigma(z)|\,d\mu(z)$$
$$= \sum_{y\in Y_i}\mu(y+V_i)\int_{V_i}|a(i,y)\chi_\sigma(y)^{-1}-\chi_\sigma(t)|\,d\mu_i(t).$$

Since the $a(i,y)$ are chosen to make $\|f_i-\chi_\sigma\|_1$ least possible,

$$\int_{V_i}|a(i,y)\chi_\sigma(y)^{-1}-\chi_\sigma(t)|\,d\mu_i(t)$$

is independent of $y\in Y_i$ and we can assume, without altering $\|f_i-\chi_\sigma\|_1$, that $a(i,y)=\chi_\sigma(y)a_i$ where $a_i=a(i,0)$. Thus $f_i=a_i\sum\chi_\sigma(y)c(i,y)$.

As $\|f_m-f_n\|_1\geqq|\,|a_m|-|a_n|\,|$ we see that $A=\lim|a_i|$ exists. If $A=0$ then $\chi_\sigma=0$ μ almost everywhere and all the a_i are 0, if $A\neq 0$ then we can replace a_i by $Aa_i|a_i|^{-1}$ and so assume $|a_i|$ constant.

We have

$$\|f_n-f_{n-1}\|_1 = \sum_{x\in X_n}\sum_{y\in Y_{n-1}}|a_n\chi_\sigma(x+y)-a_{n-1}\chi_\sigma(y)|\mu(x+y+V_n)$$
$$= M_n\sum_j(m_n-j)|a_n\chi_\sigma(x_{nj})-a_{n-1}|$$

where however $\|f_n-f_{n-1}\|_1\to 0$ as $n\to\infty$. We have now motivated and largely proved

2.1. *If $M_n\sum_j(m_n-j)|b_n\chi_\sigma(x_{nj})-b_{n-1}|\to 0$ where $\{b_n\}$ is a sequence of complex numbers with $|b_n|$ constant and $\neq 0$ implies χ_σ on S is the restriction of some $\Psi\in G^\wedge$ then μ satisfies* (i) *of* 1.1.

Proof. We know that $f_i\to\chi_\mu$ in $L^1(\mu)$. If $|a_i|=0$ for all i then $\chi_\mu=0$ and certainly $\chi_\mu=a\Psi$ for some $a\in C$, $\Psi\in G^\wedge$. If $|a_i|\neq 0$ then the condition in the theorem is satisfied with $b_i=a_i$ and there is $\Psi\in G^\wedge$ with $\chi_\sigma(s)=\Psi(s)$ for all $s\in S$. Since Ψ is continuous $a_i^{-1}f_i=\sum\Psi(y)c(i,y)\to\Psi$ uniformly on V_0 and thus in $L^1(\mu)$. Hence $a_i\Psi\to\chi_\mu$ in $L^1(\mu)$, $a=\lim a_i$ exists and $\chi_\mu=a\Psi$.

We now consider how (ii) of 1.1 can be satisfied.

2.2. (i) $D\subset[\hat\mu(G^\wedge)]^-$.

(ii) *If there is a net χ_α in $G^\wedge$ such that $\hat\mu(\chi_\alpha)\to a$ and $\chi_\alpha(s)\to 1$ for all $s\in S$ then $a\in D$ ($\hat\mu$ is the Fourier transform of μ).*

Proof. If $a\in D$ there is a net χ_α in $G^\wedge$ with $\chi_\alpha\to \boldsymbol{a}$ in $\sigma(L^\infty(\mu),L^1(\mu))$. In particular $\hat\mu(\chi_\alpha)=\int_G\chi_\alpha\,d\mu\to\int_G \boldsymbol{a}\,d\mu=a$ so that $a\in[\hat\mu(G^\wedge)]^-$.

Since if a is as in (ii) then $|a|\leqq 1$ (because $|\hat\mu|\leqq\|\mu\|=1$) and the functions

χ_α, $\boldsymbol{a}$ lie in the unit ball in $L^\infty(\mu)$ and since the span of the $c(i, y)$ is dense in $L^1(\mu)$, it is enough to show that

$$\int_G c(i, y)\chi_\alpha \, d\mu \to \int_G c(i, y)\boldsymbol{a} \, d\mu = \mu(y+V_i)a$$

for each $i = 1, 2, \ldots, y \in Y_i$. However,

$$\int \chi_\alpha c(n, y) \, d\mu = \mu(y+V_n)\chi_\alpha(y) \int \chi_\alpha \, d\mu_n$$

and

$$\int \chi_\alpha \, d\mu = \left(\int \chi_\alpha \, d\mu_n\right)\left(\sum_{y \in Y_n} \chi_\alpha(y)\mu(y+V_n)\right).$$

Since $\int \chi_\alpha \, d\mu \to a$ and the second factor in the second equation $\to 1$ for each n we have $\int \chi_\alpha \, d\mu_n \to a$ for each n and so $\int \chi_\alpha c(n, y) \, d\mu \to a\mu(y+V_n)$.

3. **The case of an infinite product of cyclic groups.** Let $\boldsymbol{Z}(n)$ denote the additive group of integers modulo n and let m_i be a sequence of integers $\geqq 2$. We shall show how the considerations of the previous section apply to $G = \boldsymbol{Z}(m_1) \times \boldsymbol{Z}(m_2) \times \cdots$. We put

$$V_i = \{g; g \in G, g_j = 0 \text{ for } j \leqq i\}$$

and take for x_{ij} that element g of G with $g_k = 0$ for $k \neq i$ and $g_i = j$.

3.1. *The condition in* 2.1 *applies to G.*

Proof. In this case $X_n \approx \boldsymbol{Z}(m_n)$ and so $\chi_\sigma(x_{nj}) = w^j$ for some m_nth root w of 1. We are thus concerned with

$$K_n = M_n \sum_j (m_n - j)|b_n w^j - b_{n-1}|.$$

If $w \neq 1$, whatever the value of b_{n-1} we have $|b_n w^j - b_{n-1}| \geqq \frac{1}{2}|b_n|$ for at least half the terms in the sum and the sum of at least half the terms in the sequence $M_n m_n, M_n(m_n-1), \ldots, M_n \cdot 1$ is at least $\frac{1}{4}$. Thus $K_n \geqq \frac{1}{8}|b_n|$ and if $K_n \to 0$ we must have $w = 1$ for large values of n and the result follows.

3.2. *G satisfies hypothesis* (ii) *of* 1.1.

Proof. The values of $\hat{\mu}$ are 1 and finite products of numbers of the form

$$2m^{-1}(m+1)^{-1} \sum_{j=0}^{m-1} (m-j)w^j = 2(m+1)^{-1}(1-w)^{-1}$$

where w is an mth root of 1 different from 1. Since any number of this form is < 1 in absolute value and $|2(m+1)^{-1}(1-w)^{-1}| \leqq (m+1)^{-1}(\sin \pi/m)^{-1} \to \pi^{-1}$ as $m \to \infty$ we see that the values of $|\hat{\mu}|$ are not dense in (0, 1). Thus by (i) of 2.2, the second part of (ii) of 1.1 is satisfied.

By what we have shown above the sequence $\Omega_j = 2(m_j+1)^{-1}[1-\exp(2\pi i m_j^{-1})]^{-1}$ has a limiting point a with $0<|a|<1$. Suppose Ω_{j_k} is a subsequence with $\Omega_{j_k} \to a$ and consider the character $\chi_k(g) = \exp(2\pi i g_{j_k} m_{j_k}^{-1})$. Then for large k, $\chi_k = 1$ on X_n and so certainly $\chi_k(s) \to 1$ on S and $\hat{\mu}(\chi_k) = \Omega_{j_k} \to a$ so that the first part of 1.1 (ii) follows from 2.2 (ii). We have now proved

3.3. $Z(m_1) \times Z(m_2) \times \cdots$ *satisfies the hypotheses of* 1.1.

4. **The case of the p-adic integers.** Throughout this section p is a fixed prime number. The discrete group p^∞ of all p^nth ($n=1, 2, \ldots$) roots of unity has as dual the additive group I_p of the p-adic integers which is a compact abelian topological group. The elements of I_p can be denoted by infinite sequences $(k_1, k_2, \ldots)$ of integers with $k_{n+1} = k_n \bmod p^n$ (see [1, p. 154]). Two such sequences (k_n), (l_n) with $k_n = l_n \bmod p^n$ for all n determine the same element of I_p. The duality between p^∞ and I_p is given by

$$\langle \exp(2\pi i m p^{-n}), (k_1, k_2, \ldots)\rangle = \exp(2\pi i m k_n p^{-n}).$$

The elements of I_p can also be denoted by sequences $[\kappa_1, \kappa_2, \ldots]$ with $\kappa_i = 0, 1, \ldots, p-1$ where $k_i = \kappa_1 + p\kappa_2 + \cdots + p^{i-1}\kappa_i$. The topology on I_p is that inherited in this way from a countable product of discrete p element spaces.

To apply the considerations of §2 we take $m_i = p$ for all i, $V_i = p^i I_p (= \{p^i g; g \in I_p\})$. For $n \in \boldsymbol{Z}$, the group of integers, let $\boldsymbol{n} = (n, n, n, \ldots)$. Then $n \to \boldsymbol{n}$ is a group homomorphism of $\boldsymbol{Z}$ into I_p. We put $x_{ij} = \boldsymbol{n}$ where $n = jp^{i-1}$. S is the image of $\boldsymbol{Z}$ under $n \to \boldsymbol{n}$ and, since χ_σ is a character on S, we have $\chi_\sigma(\boldsymbol{n}) = \exp 2\pi i n\alpha$ for some $\alpha \in [0, 1)$.

4.1. *The condition in* 2.1 *applies in* I_p.

Proof. We are concerned with

$$2p^{-1}(p+1)^{-1}[p|b_n - b_{n-1}| + (p-1)|b_n \exp 2\pi i \alpha p^{n-1} - b_{n-1}| + \cdots].$$

Since this tends to 0 and is a sum of positive terms we have $b_n - b_{n-1} \to 0$ and $b_n \exp 2\pi i\alpha p^{n-1} - b_{n-1} \to 0$. Subtracting we get $b_n(1-\exp 2\pi i \alpha p^{n-1}) \to 0$ and since $|b_n|$ is a nonzero constant $\exp 2\pi i\alpha p^{n-1} \to 1$ as $n \to \infty$. This can only occur if $\alpha = kp^{-l}$ for some integers k, l. However, this implies that χ_σ on S is the restriction of the continuous character corresponding to $\exp 2\pi i k p^{-l} \in p^\infty$.

4.2. *Condition* (ii) *of* 1.1 *is satisfied in* I_p.

Proof. For $\exp 2\pi i k p^{-l} \in p^\infty$, $0<k<p^{-l}$, the value of $\hat{\mu}$ at the corresponding character is

$$\prod_{m=0}^{l-1} 2p^{-1}(p+1)^{-1} \sum_{j=0}^{p-1} (p-j) \exp 2\pi i k j p^{m-l}.$$

As in 3.2 the absolute values of such products are bounded away from 1 and the second part of (ii) of 1.1 is satisfied.

Now let χ_l be the character corresponding to the element $\exp 2\pi i p^{-l}$ of p^∞. $\chi_l(\boldsymbol{n}) = \exp 2\pi i n p^{-l} \to 1$ as $l \to \infty$ and $\hat{\mu}(\chi_l)$ tends to

$$\prod_{m=0}^{\infty} 2p^{-1}(p+1)^{-1} \sum_{j=0}^{p-1} (p-j) \exp 2\pi i j p^{-m}$$

the infinite product converging to a limit in $0<|z|<1$ since $\exp(2\pi i j p^{-m})-1 \sim 2\pi i j p^{-m}$ as $m \to \infty$ and $\sum p^{-m}$ converges. Thus by (ii) of 2.2 the first part of (ii) of 1.1 is satisfied.

4.3. *I_p satisfies the hypotheses of* 1.1.

5. **The cases T and R.** For T, the additive group of the reals modulo 1 we put $m_i = 2$ for all i, $V_i = [0, 2^{-i})$, $x_{i0} = 0$ and $x_{i1} = 2^{-i}$. Although the V_i are not compact neighborhoods of 0 the calculations in §2 still apply since continuous functions can still be approximated uniformly by the functions $g_i \in C_i$ and μ_n is concentrated on V_n since its support is in V_n^- and one point sets are of μ measure zero for

$$\mu([m2^{-n}, (m+1)2^{-n}]) \leqq (2/3)^n \to 0$$

as $n \to \infty$.

5.1. *The condition in* 2.1 *applies to T.*

Proof. We are concerned with

$$\tfrac{2}{3}|b_n - b_{n-1}| + \tfrac{1}{3}|w_n b_n - b_{n-1}|$$

where $w_n = \chi_\sigma(2^{-n})$ is a 2^nth root of 1. If this tends to 0 we have $w_n \to 1$ from which it follows that there is an integer l with $w_n = \exp 2\pi i l 2^{-n}$ for all n. Thus, on S, χ_σ is the restriction of $\exp 2\pi i l x \in T^\frown$.

5.2. *T satisfies* (ii) *of* 1.1.

Proof. The values of $\hat{\mu}$ are the Fourier coefficients $\hat{\mu}_l$ of μ. We have

$$\hat{\mu}_l = \prod_{n=1}^{\infty} 3^{-1}(2+\exp 2\pi i l 2^{-n})$$

which is 1 if $l=0$ and is $\leqq 1/3$ in absolute value if $l \neq 0$.

For the first part of (ii) we take $\chi_k(x) = \exp 2\pi i 2^k x$ so that $\chi_k(x) \to 1$ as $k \to \infty$ for each $x \in T$ and $\hat{\mu}(\chi_k) \to \prod_{n=1}^{\infty} 3^{-1}(2+\exp 2\pi i 2^{-n})$, a number in $0<|z|<1$.

5.3. *T satisfies the hypotheses of* 1.1.

5.4. ***R*** *satisfies the hypotheses of* 1.1.

Proof. We make the same construction as in T and the proof is similar except that l takes on all real values and 2.2(i) is not enough to give the second part of 1.1(ii). However if $a \in D$ there is a net l_α in $\boldsymbol{R}$ with the corresponding characters

$\exp 2\pi i x l_\alpha \to \boldsymbol{a}$ in $\sigma(L^\infty(\mu), L^1(\mu))$. If l_α has a bounded subnet it has a subnet convergent to some $l \in \boldsymbol{R}$ so that $a = \exp 2\pi i x l$ μ-almost everywhere and $|a| = 1$. Otherwise l_α has no bounded subnet and since for $2^k \leqq |l_\alpha| \leqq 2^{k+1}$

$$\left|\int \exp 2\pi i l_\alpha x \, d\mu(x)\right| \leqq \tfrac{1}{3}\,|2+\exp 2\pi i l_\alpha 2^{-k-2}| \leqq \sqrt{5}/3$$

we see that $|a| \leqq \sqrt{5}/3$ and 1.1(ii) easily follows.

6. Results in subgroups and quotient groups.

6.1. *If G is a* L.C.A. *group and H is a closed subgroup with a $\mu \in M(H)$ satisfying the conditions of* 1.1 *then μ, considered as a measure on G, satisfies the conditions of* 1.1.

Proof. The result follows from the fact that $G \backslash H$ is of μ measure zero, that the continuous characters on H are exactly the restrictions of the continuous characters on G [3, p. 36] and the restriction of a generalized character on G to H is a generalized character on H.

6.2. *If G is a* L.C.A. *group and H a compact subgroup such that $\partial M \neq \Delta$ for G/H then $\partial M \neq \Delta$ for G.*

Proof. If $\mathfrak{A}$ is a commutative Banach algebra with unit u and e is an idempotent in $\mathfrak{A}$ then $\mathfrak{A}$ is the direct sum of the ideals $e\mathfrak{A}$ and $(u-e)\mathfrak{A}$ and the maximal ideal space of $\mathfrak{A}$ is the union of the maximal ideal spaces of $e\mathfrak{A}$ and $(u-e)\mathfrak{A}$. The Šilov boundary of $\mathfrak{A}$ is the union of the Šilov boundaries of $e\mathfrak{A}$ and $(u-e)\mathfrak{A}$ so that if the Šilov boundary of $e\mathfrak{A}$ is not the whole of the maximal ideal space of $e\mathfrak{A}$ the same is true for $\mathfrak{A}$.

If we have $\mathfrak{A} = M(G)$, $e = \lambda_H$, the Haar measure of H then the ideal $e\mathfrak{A}$ is isomorphic with $M(G/H)$ [3, §2.7] so that this result follows from the general considerations above.

7. The main result.

7.1. *Let G be a compact nondiscrete group. Then G has a closed subgroup H such that G/H satisfies* 1.1.

Proof. By [3, p. 35] it is enough to show that $G^\wedge$ has a subgroup which is the dual of a group satisfying 1.1. By 3.3, 4.3 and 5.3 this will happen if $G^\wedge$ contains a direct sum of an infinity of cyclic groups, if $G^\wedge$ contains a group p^∞ or if $G^\wedge$ contains an element of infinite order. Since $G^\wedge$ is infinite using the results on pp. 137, 164 and 165 of [1] the only case in which one at least of these possibilities does not hold is the case in which $G^\wedge$ is the direct sum of a finite number of reduced primary groups and so contains a countable reduced primary group Π. Let Π_1 be the group of elements of infinite height in Π so that, as in the construction of Ulm factors [1, p. 189], Π/Π_1 is a direct sum of a sequence of cyclic groups, thus by 3.3 $(\Pi/\Pi_1)^\wedge$ satisfies 1.1 and so, by 6.1, does $\Pi^\wedge$.

7.2. THEOREM. *Let G be a nondiscrete locally compact abelian group. Then* $\partial M \neq \Delta$.

Proof. By [3, Theorem 2.4.1] G has an open closed subgroup G_1 which is the direct sum of a compact group and a copy of R^n. If $n>0$ the present result follows from 5.4, 6.1 and 1.1. If $n=0$ then G_1 is compact and nondiscrete and so by 7.1 contains a compact subgroup H such that G_1/H satisfies 1.1. By 6.1 G/H satisfies 1.1 and by 1.1 and 6.2 the result follows.

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